$$x_1=OP+PA=a+a\cos\alpha=a(1+\cos\alpha)\,;\ i.\ e.,\ OD=\tfrac{1}{2}x_1=\frac{a}{2}(1+\cos\alpha),$$

which gives as the polar equation of the locus referred to $O$ as origin and $OP$ as initial line,

$$\rho=\frac{a}{2}(1+\cos\alpha),$$

a cardioid, with cusp at $O$ passing through $P$ and having $OP$ as line of symmetry.

Also solved by J. Scheffer, A. H. Holmes, V. M. Spunar, G. B. M. Zerr, and J. W. Clawson.

## CALCULUS.

**264. Proposed by W. J. GREENSTREET, M. A., Stroud, England.**

The join of the center of curvature of a curve to the origin is at $\alpha$ to the initial line. Prove that with the usual notation:

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2+\left(\frac{d^2p}{d\psi^2}\right)^2\right]=\frac{dp}{d\psi}.\frac{d\rho}{d\psi}.$$

**Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.**

Denote the coordinates of the center of curvature by $\xi$, $\eta$. Then $\xi=x-\rho\sin\phi$, $\eta=y+\rho\cos\phi$, $\rho$ being the radius of curvature and $\phi$ the angle formed by the tangent and positive $x$-axis.

$\therefore \tan\alpha=\eta/\xi$, and $d\alpha=\dfrac{\xi\,d\eta-\eta\,d\xi}{\xi^2+\eta^2}$.

Likewise, $\tan\psi=r\dfrac{d\theta}{dr}$, and $d\psi=\left[d\theta+r\left(\dfrac{d^2\theta}{dr^2}\right)\right]\div\sec^2\psi$, where $\psi$=angle included by the tangent at $(r, \theta)$, and the radius vector to the point $(r, \theta)$.

Substituting the proper values of $\xi$, $\eta$, $d\xi$, $d\eta$ in $d\alpha$, expressed in polar coordinates, and we have the first product $d\alpha/d\psi$.

Also, on remembering that $\left\{\begin{matrix} p=r\sin\psi \\ \varphi=\theta+\psi\end{matrix}\right\}$, differentiate it twice, square every differential quotient as indicated by the proposition, add the like terms, reduce, and we have, readily,

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2+\left(\frac{d^2p}{d\psi^2}\right)^2\right]=\frac{dp}{d\psi}\cdot\frac{d\rho}{d\psi}.$$

**265. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.**

Find two curves which possess the property that the tangents $TP$ and $TQ$ to the inner one always makes equal angles with the tangent $TT'$ to the outer.

**Remarks by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

In Salmon's *Conic Sections*, Sixth Edition, Articles 188-189, it is demonstrated that if through any point $P$ of a conic section we draw tangents $PT$, $QT$ to a confocal conic section, these tangents are equally inclined to the tangent at $P$. For every point $P$ the outer conic through $P$, though confocal to the inner, has different axes.

If we wish to find a curve for all positions of $P$ the problem assumes some difficulty.

For the ellipse, the equation to $TP$, $TQ$ is

$$(a^2-h^2)(y-k)^2+2(y-k)(x-h)hk+(b^2-k^2)(x-h)^2=0,$$

where $(h, k)$ are the coordinates of $T$.

$$2hk(y-k)+\{b^2-a^2+h^2-k^2\pm\sqrt{[(b^2-a^2+h^2-k^2)^2+4h^2k^2]}\}(x-h)=0,$$

gives the two lines making equal angles with $TP$, $TQ$.

The envelope of the line formed by using the plus sign subject to the condition obtained by using the minus sign gives the required curve.

If $a=b$, the ellipse becomes a circle. Then $TT'$ is

$$ky+hx=h^2+k^2 \dots (1).$$

$hy=kx$ is the perpendicular to (1).

The envelope of (1) subject to the condition $hy=kx$ ,is $x^2+y^2=0$, or the center of the given circle.

**266. Proposed by C. N. SCHMALL, New York City.**

Show that the $n$th derivative of the fraction $u/v$ can be expressed in the form of a determinant, $u$ and $v$ being functions of $x$.

**Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.**

As $u=\phi(x)$, $v=\psi(x)$, hence, $F(x)=\phi(x)/\psi(x)=u/v$, and therefore,

$$F'(x)=\frac{vu'-uv'}{v^2}=\frac{1}{v^2}\begin{vmatrix} v & v' \\ u & u' \end{vmatrix}=\frac{u_1}{v_1}=\frac{\phi_1(x)}{\psi_1(x)}.$$

Likewise, $F''(x)=\dfrac{1}{v_1^2}\begin{vmatrix} v_1 & v'_1 \\ u_1 & u'_1 \end{vmatrix}$; $F'''(x)=\dfrac{1}{v_2^2}\begin{vmatrix} v_2 & v'_2 \\ u_2 & u'_2 \end{vmatrix}$; ...

$\therefore F^{(n)}(x)=\dfrac{1}{v^2_{n-1}}\begin{vmatrix} v_{n-1} & v'_{n-1} \\ u_{n-1} & u'_{n-1} \end{vmatrix}$, where $v_\lambda=v^{2^\lambda}$, $v'_\lambda=2^\lambda v^{2^\lambda-1}v'$;